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# D-branes in unoriented non-critical strings and duality in $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ gauge theories

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Abstract: We exhibit exact conformal field theory descriptions of $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ pairs of Seiberg-dual gauge theories within string theory. The $\mathcal{N} = 1$ gauge theories with flavour are realized as low energy limits of the worldvolume theories on D-branes in unoriented non-critical superstring backgrounds. These unoriented backgrounds are obtained by constructing exact crosscap states in the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset conformal field theory using the modular bootstrap method. Seiberg duality is understood by studying the behaviour of the boundary and crosscap states under monodromy in the closed string parameter space.

Keywords: Brane Dynamics in Gauge Theories, Conformal Field Models in String Theory, Gauge-gravity correspondence.

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## 1. Introduction

Electric-magnetic duality of $\mathcal{N} = 1$ gauge theories in four dimensions is an important tool to understand supersymmetric generalizations of quantum chromodynamics [1]. For instance, it has recently been used to argue for the *generic* existence of long-lived metastable supersymmetry breaking vacua in supersymmetric gauge theories [2]. In order to further exploit the fruitful interplay between gauge theory and string theory, it is natural to realize Seiberg duality within a brane set-up in string theory [3, 4]. To sharpen the embedding map, it is useful to incorporate the backreaction of the heaviest, NS5-branes, used within the brane set-up into the background geometry. It is possible to do this in a double scaling limit [5, 6]. The D-branes that realize the gauge theories can then be

described by an exact boundary conformal field theory, and one can analyze Seiberg duality exactly within string theory, as argued and realized in [7] for supersymmetric QCD with $\mathrm{SU}(N_c)$ gauge group. It is worth trying to extend this to $SO$ and $Sp$ gauge theories, since both the bulk and boundary conformal field theories, as well as the gauge theories, realize new physical phenomena.

In the rest of the introduction we review Seiberg duality in $SO/Sp$ gauge theories [8] in the context of NS5/D-brane set-ups in type IIA string theory, in as far as it is useful for our present purposes. We then move to a worldsheet description of the backreacted theory in a double scaling limit in section 2. The main technical result in this section is the computation of the supersymmetric crosscap states built on the representations of the $\mathcal{N} = 2$ superconformal algebra. In the modular bootstrap procedure used to construct D-branes in Liouville theory [9, 10] and their supersymmetric extensions, [11 – 14], the crosscap states are found by requiring that, in the overlap of the crosscaps with the localized brane, the $\Omega$-twisted characters of the superconformal algebra are obtained in the open string channel. We give a closed string orientifold description of the different crosscaps by computing the Klein-bottle amplitudes. Some technical details regarding the amplitudes are collected in the appendices. Crosscap states in the cigar background have already been studied in detail in a "continuum limit" [15]. However, for our purposes, it is necessary to require integral $\mathrm{U}(1)_R$ charges so as to be able to GSO project.

In section 3, we use these crosscaps along with the boundary states described in [16, 17] to engineer a $SO/Sp$ gauge theory in four dimensions with flavours in the fundamental representation. We study how electric-magnetic or Seiberg duality is realized in the gauge theory from a worldsheet perspective, following the ideas in [7]. This involves understanding how boundary states and crosscaps behave under a monodromy in the closed string parameter space. We find that within the context of a smooth superstring background, our results realize the results obtained in the geometric type IIA set-up involving D4 branes and NS5 branes [4] and are consistent with field theory expectations. We conclude with a brief discussion in section 4.

### 1.1 Seiberg duality from NS5/D-brane set-ups

When tackling the exact description of Seiberg duality in the double scaling limit, it will be useful to keep in mind the bulk configuration before taking the limit. We review briefly how Seiberg duality was analyzed by studying the low energy theory on D4 branes suspended between the NS5 branes in type IIA string theory. The realization of Seiberg duality in this brane set up has been discussed in detail in the literature [4, 18, 19]. As we will show, some aspects of this description of Seiberg duality are closely related to our worldsheet discussion.

We begin with the electric set-up as shown in figure 1. We have shown the position of the various objects in the $x^6$ direction. The O4 plane extends all along the $x^6$ direction. Note that when the O4 plane crosses an NS5 brane, it goes from the positively charged + type to the negatively charged − type.

Flavour is realized by D6 branes to the right of the NS brane as shown in figure 1. Since the low-energy gauge theory physics is independent of the position of the D6 brane,



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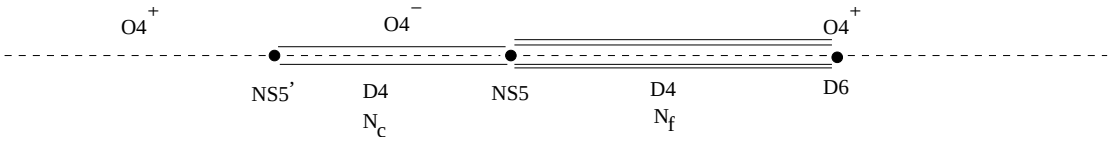


**Figure 1:** Electric brane set-up, including an orientifold four-plane that changes type, and $N_c$ colour D4-branes and $N_f$ flavour D4-branes ending on D6-branes.

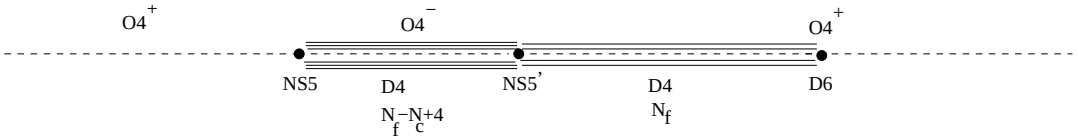


**Figure 2:** Magnetic brane set-up, including an orientifold four-plane that changes type, and $N_f - N_c + 4$ colour D4-branes and $N_f$ flavour D4-branes ending on D6-branes.

one can also take the limit when it is at infinity. This limit has an interesting relation to the geometric engineering of gauge theories using D6 branes wrapped on the 3-sphere [20, 21] in the deformed conifold. Seiberg duality in that context was implemented by the so-called $\mu-$transition. The deformed conifold is defined as the zero locus

$$x^2 + y^2 + z^2 + u^2 = \mu \qquad \text{in} \quad \mathbb{C}^4 \,. \tag{1.1}$$

The $\mu-$transition amounts to the operation $\mu \to -\mu$. In the absence of flavour branes, such transitions have been studied in detail in [22] by lifting the brane and orientifolds to M-theory following earlier work in [23].

Now the NS5 brane configuration we started with is related to the geometric set-up by T-dualities [24, 6]. It is known that the closed string parameter $\mu$ of the conifold maps to the relative position of the NS and NS' branes. Electric-magnetic duality is therefore implemented in the IIA brane set-up by exchanging the positions of the NS' and NS along the $x^6$ direction. The presence of the orientifold implies that we necessarily pass through a strong coupling region in order to implement Seiberg duality [19].

In order to derive the magnetic configuration, one uses the fact that a certain linking number [3] must be conserved in the duality transition. For the NS5 brane, the linking number is defined for each NS5-brane as [3, 4]

$$l_{NS} = \frac{1}{2}(R_{D6} - L_{D6}) + (L_{D4} - R_{D4}) + Q(O4)(L_{O4} - R_{O4}) \,, \tag{1.2}$$

where $R(X)$ $(L(X))$ refer to the number of branes of type $X$ to the right (left) of the NS5 brane. For the electric configuration, which leads to a SO($N_c$) theory with $N_f$ flavours, the linking number for the NS brane is equal to

$$l_{NS} = -\frac{1}{2}\, N_f + N_c - 2 \,. \tag{1.3}$$

We count charges such that $N_c$ D4 branes lead to a colour group of SO($N_c$). We have shown the magnetic configuration in figure 2.

Note that we are forced to include an additional 4 D4 branes in order to leave invariant the linking number. One can check that this brane configuration has the correct properties of the Seiberg dual gauge theory, including the presence of the $\frac{1}{2}N_f(N_f+1)$ mesonic fields (the fluctuation of the $D4$ branes stretched between the NS' and D6) and the superpotential term $Mq\tilde{q}$.

It is the equality of the linking number in the electric and magnetic configurations that forces us to include the 4 extra D4 branes that realize the colour gauge theory. It is not possible to derive the generation of the 4 extra D4 branes in this semi-classical brane set-up. See however [19] for a derivation of this phenomenon from a strong coupling analysis.

Our analysis of the IIA brane set-ups has been brief as these set-ups are fairly well studied in the literature. We now turn to describing the bulk geometry and the associated branes exactly in terms of conformal field theory, in a suitable limit of the NS5 brane background.

## 2. Worldsheet description of the brane/orientifold system

For type IIA strings on the deformed conifold, one can define a double scaling limit

$$\mu \to 0 \qquad g_s \to 0 \qquad \text{keeping} \qquad \frac{\mu}{g_s} \qquad \text{fixed}\,. \tag{2.1}$$

The mass of the D-branes that wrap the three-sphere remains finite. In this limit, it is conjectured that the decoupled non-gravitational theory near the singularity has a holographic description as a string theory whose worldsheet description is the conformal field theory [5, 6]

$$\mathbb{R}^{1,3} \times \left[\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\right]_{k=1}\,, \tag{2.2}$$

where the coset conformal field theory is at supersymmetric level $k=1$. We note in passing that the relation between D-branes in NS5 backgrounds and the geometric engineering of gauge theories using branes on the deformed conifold, that was mentioned earlier, is consistent with this conjecture. The topological sub-sector of the CFT (2.2) has been matched to the topological sector of the deformed conifold theory for both closed strings [25, 26] and open strings [27].

One can realize SQCD like theories in this set-up by introducing $N_c$ D3-branes localized at the tip of the cigar associated to an $\mathrm{U}(N_c)$ gauge group [16, 17] and $N_f$ double-sheeted D5-branes filling the whole of space-time that introduce $N_f$ flavors into the gauge theory [16, 7]. We next turn to a detailed discussion of the crosscaps and boundary states in this background that will allow us to understand Seiberg duality from a worldsheet perspective.

### 2.1 Review of cylinder amplitudes and boundary states

In [7] it was argued that a sign flip of the cigar bulk interaction coefficient $\mu$ leads to an exact conformal field theory realization of Seiberg duality for $\mathrm{U}(N_c)$ gauge theories. We recall that from the cylinder amplitudes for these branes, one could derive the one-point

functions via a modular bootstrap procedure [10, 11, 14]. In this way one could associate to characters of the $N = 2$ superconformal algebra (labeled by $J, M$, functions of the conformal dimension and R-charge) particular Cardy boundary states. A further crucial ingredient in [7] was the addition relation:

$$|D5, J = 0\rangle = |D3\rangle + \overline{|D5, J = \frac{1}{2}\rangle}\,. \tag{2.3}$$

for exact boundary states. This is a consequence of an $N = 2$ superconformal algebra character identity at level $k = 1$, relating the $J = 0, M = 0$ continuous character, with the $J = 1/2, M = 1/2$ continuous character and the identity character.

We briefly recall a few annulus amplitudes for D-branes in the background (2.2), as discussed in [11, 14, 16, 17, 7]. By a $D3$ brane in the background (2.2), we mean a D-brane that fills $\mathbb{R}^{1,3}$ that is tensored with the identity brane in the cigar direction. We denote the corresponding boundary state as $|D3\rangle$ as above. Similarly, a $D5$ brane fills $\mathbb{R}^{1,3}$ and is tensored with a brane labeled by the continuous representation $|D5, J, M\rangle$. We mostly focus on the branes with $J = M = 0$ and $J = M = \frac{1}{2}$ and will neglect the $M$ eigenvalue.

The modular bootstrap is implemented by requiring that only the identity character appears in the self-overlap of the $D3$ brane while the continuous character appears in the overlap of the $D3$ and the $D5$ branes. In what follows, we will use the notation $q_o = e^{-2\pi t}$ and $q_c = e^{-2\pi s}$ for the open and closed string channel moduli. In the open string channel, the cylinder amplitude for the overlap of the $J = \frac{1}{2}$ D5 brane with the D3 brane is given by

$$B\left(D3; J=\frac{1}{2}\right) = \frac{1}{2}\int\frac{dt}{2t}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-2\pi tk^2}}{\eta(it)^6}[\Theta_{1,1}(it)\vartheta_{00}^2(it)+\Theta_{1,1}(it)\vartheta_{01}^2(it)-\Theta_{0,1}(it)\vartheta_{10}^2(it)]\,. \tag{2.4}$$

Similarly for the $J = 0$ overlap with the D3, we get

$$B(D3; J = 0) = \frac{1}{2}\int\frac{dt}{2t}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-2\pi tk^2}\,q_o^{-1/4}}{\eta(it)^6}\left[\Theta_{0,1}(it)\vartheta_{00}^2(it) - \Theta_{0,1}(it)\vartheta_{01}^2(it) - \Theta_{1,1}(it)\vartheta_{10}^2(it)\right]\,. \tag{2.5}$$

Using the identities in equation (A.4) of the appendix, one can show that both these amplitudes vanish. All three boundary states are therefore mutually supersymmetric.

From the identity in (2.3), we see that from the difference of (2.5) and (2.4), one obtains the D3 self-overlap. Indeed, the $J = 0$ representation of the $N = 2$ superconformal algebra is reducible. Using the modular bootstrap method, one can modular transform the annulus amplitude into the closed string channel and read off the D3 and D5 brane wavefunctions for both $J = 0$ and $J = \frac{1}{2}$. This has been done in [11, 14] and we record the

NS sector wavefunctions below:[1]

$$\Psi^{\mathrm{NS},\pm}_{\mathrm{Id}}(P,w) = \frac{\mathcal{C}_\pm}{\sqrt{2}}\,\frac{\Gamma(\frac{1}{2}+iP+\frac{w}{2})\Gamma(\frac{1}{2}+iP-\frac{w}{2})}{\Gamma(2iP)\Gamma(1+2iP)}$$
$$\Psi^{\mathrm{NS},\pm}_{\mathrm{cont},J=\frac{1}{2}}(P,w) = 2\,\frac{\mathcal{C}_\pm}{\sqrt{2}}\,,\frac{\Gamma(-2iP)\Gamma(1-2iP)}{\Gamma(\frac{1}{2}-iP+\frac{w}{2})\Gamma(\frac{1}{2}-iP-\frac{w}{2})}$$
$$\Psi^{\mathrm{NS},\pm}_{\mathrm{cont},J=0}(P,w) = 2\,\frac{\mathcal{C}_\mp}{\sqrt{2}}\,\cosh 2\pi P\,\frac{\Gamma(-2iP)\Gamma(1-2iP)}{\Gamma(\frac{1}{2}-iP+\frac{w}{2})\Gamma(\frac{1}{2}-iP-\frac{w}{2})}\,, \tag{2.6}$$

where $\mathcal{C}_+ = 1$ and $\mathcal{C}_- = e^{i\pi w}$. Similar expressions also exist for the RR sectors that can be easily obtained from the expression above by spectral flow $w \to w+1$.

### 2.2 Möbius amplitudes and crosscap states

To construct crosscap states, we proceed in analogy with the boundary states. We will perform a modular bootstrap procedure for the Möbius amplitudes for the supercoset conformal field theory at level $k = 1$. It is clear how to generalize our analysis to other levels, but we will not need the generalization for the applications in this paper. Our starting point will be the requirement that our crosscap states have an overlap with the identity D3-brane which is an $\Omega$-twisted $N = 2$ superconformal character (as in [15]). We then use the open-closed duality of the Möbius amplitudes and our knowledge of the D3-brane boundary state to derive corresponding crosscap states. Afterwards, we will interpret the crosscap states in terms of orientifolds of the bulk superstring theory by computing the Klein-bottle amplitudes.

Twisting ordinary $N = 2$ superconformal characters in the open string channel by $\Omega$ can be implemented by the substitution $\tau \to \tau + \frac{1}{2}$ in the untwisted characters $\chi(\tau)$. One obtains the twisted characters $\chi_\Omega(\tau)$ that take into account the action of $\Omega$ on the fermionic oscillators. The full Möbius amplitude that corresponds to the overlap of the crosscap state labeled by the $J = \frac{1}{2}$ continuous character with the D3 brane (and vice versa) is then given by

$$M_\Omega\Big(D3; J=\frac{1}{2}\Big) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-2\pi t k^2}}{\eta(it+\frac{1}{2})^6}\left[\Theta_{1,1}\Big(it+\frac{1}{2}\Big)\left(\vartheta_{00}^2\Big(it+\frac{1}{2}\Big)+\vartheta_{01}^2\Big(it+\frac{1}{2}\Big)\right)\right.$$
$$\left.-\Theta_{0,1}(it+\frac{1}{2})\vartheta_{10}^2\Big(it+\frac{1}{2}\Big)\right]\,. \tag{2.7}$$

The Möbius amplitude for the overlap of the $J = 0$ crosscap state with the D3 brane can similarly be written as[2]

$$M_\Omega(D3; J=0) = \frac{e^{-\frac{i\pi}{4}}}{2}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-2\pi t k^2}\,q_o^{-1/4}}{\eta(it+\frac{1}{2})^6}\left[\Theta_{0,1}\Big(it+\frac{1}{2}\Big)\left(\vartheta_{00}^2\Big(it+\frac{1}{2}\Big)-\vartheta_{01}^2\Big(it+\frac{1}{2}\Big)\right)\right.$$
$$\left.-\Theta_{1,1}\Big(it+\frac{1}{2}\Big)\vartheta_{10}^2\Big(it+\frac{1}{2}\Big)\right] \tag{2.8}$$

[1] We suppress the $\mu$-dependent phase factor in all wavefunctions (both for the boundary states and the crosscaps to follow). We will come back to this point in section 4.

[2] The factor of $e^{-\frac{i\pi}{4}}$ ensures that the small $q$ expansion of the characters begins with a real number.

Using the same theta function identities that were used to show the vanishing of the D5-D3 overlap (for $\tau' = \tau + \frac{1}{2}$), we see that both the overlaps vanish. Thus both these crosscap states are supersymmetric with respect to the D3 brane.

Similar to the D-brane boundary states, we define the crosscap state corresponding to the $\Omega$-twisted identity character as the difference between the $J = 0$ and $J = \frac{1}{2}$ crosscap state. Let us now execute the modular bootstrap and determine the closed string one-point functions explicitly.

**Modular bootstrap for the crosscap**

Modular transforming the first term of the $J = \frac{1}{2}$ Möbius amplitude into the closed string channel, and using that the closed string and open string channel parameters are now related by $s = \frac{1}{4t}$, we get

$$M_{\Omega}^{\rm NS}\left(D3; J = \frac{1}{2}\right) = 8e^{\frac{i\pi}{2}}\,\frac{1}{2}\int \frac{ds}{128\pi^4}\,\frac{e^{\frac{i\pi}{8}}\,\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\int_0^\infty dP\,2\,\vartheta_{01}(2is)\,q_c^{P^2}\,\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3} \tag{2.9}$$

The details of the modular transformation are given in the appendix. For the $J = 0$ Möbius amplitude in equation (2.8), the $q_o^{-\frac{1}{4}}$ factor is first transformed into

$$e^{\frac{\pi}{8s}} = \sqrt{2s}\int_0^\infty dP\,\cosh\pi P\,e^{-2\pi s P^2}\,. \tag{2.10}$$

For the NS sector, we get for the $J = 0$ Möbius amplitude

$$M_{\Omega}^{\rm NS}(D3; J = 0) = 4\int \frac{ds}{128\pi^4}\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\int_0^\infty dP 2\cosh\pi P\vartheta_{10}(2is)q_c^{P^2}\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}. \tag{2.11}$$

Following the modular bootstrap approach that led to the boundary states, one can use the results for the D-brane wavefunctions in equation (2.6) to derive the wavefunctions for the crosscap states.

Let us begin with the $J = \frac{1}{2}$ overlap in equation (2.9): expanding the theta-function $\vartheta_{01}(2is)$ as we did before, we get

$$M_{\Omega}^{\rm NS}(D3; J = \frac{1}{2}) = e^{\frac{i\pi}{2}}\int \frac{ds}{32\pi^4}\,\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\sum_{n\in\mathbb{Z}}(-1)^n\int_0^\infty dP\,2\,q_c^{P^2+n^2}\,\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\,. \tag{2.12}$$

This should be rewritten in terms of the closed string wavefunctions as

$$2\int \frac{ds}{128\pi^4}\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\sum_{n\in\mathbb{Z}}\int_0^\infty dP\,\Psi_{\rm Id}^{{\rm NS},\mp}(-P,2n)\,\Psi_{C,{\rm cont},J=\frac{1}{2}}^{{\rm NS},\pm}(P,2n)\,ch_{c,\Omega}^{\tilde{\rm NS}}(P,2n,is)\,. \tag{2.13}$$

Here, the $\Omega$-twisted character $ch^{\tilde{\rm NS}}_{c,\Omega}$ is given by a twisted overlap of the Ishibashi states

$$\mathrm{ch}^{\tilde{\rm NS}}_{c,\Omega}(P,n,is) = q_c^{P^2+\frac{n^2}{4}}\,\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\,. \tag{2.14}$$

Note that in the overlap, only the even winding modes $w=2n$ appear. The crosscap state can therefore be written as a linear combination of Ishibashi states with even winding:

$$|C,J=\frac{1}{2};\pm\rangle_{\rm NSNS} = \sum_{n\in\mathbb{Z}}\int_0^\infty dP\,\Psi^{\rm NS,\pm}_{C,{\rm cont},J=\frac{1}{2}}(P,2n)\,i^{(L_0^{\rm osc}+\overline{L}_0^{\rm osc})}\,i^{(J_0^{\rm osc}+\overline{J}_0^{\rm osc})}\,|P,2n;\pm\rangle\rangle\,, \tag{2.15}$$

where the crosscap wavefunction is given by

$$\Psi^{\rm NS,\pm}_{C,{\rm cont},J=\frac{1}{2}}(P,2n) = \frac{8}{\sqrt{2}}\,i\,e^{i\pi n}\,\frac{\Gamma(-2iP)\Gamma(1-2iP)}{\Gamma(\frac{1}{2}-iP+n)\Gamma(\frac{1}{2}-iP-n)}\,. \tag{2.16}$$

Let us now consider the open string contribution in the RR sector. Similar manipulations yield that the crosscap state is a sum of only those Ishibashi states with odd winding modes

$$|C,J=\frac{1}{2};\pm\rangle_{\rm RR} = \sum_{n\in\mathbb{Z}}\int_0^\infty dP\,\Psi^{\rm R,\pm}_{C,{\rm cont},J=\frac{1}{2}}(P,2n-1)\,i^{(L_0^{\rm osc}+\overline{L}_0^{\rm osc})}\,i^{(J_0^{\rm osc}+\overline{J}_0^{\rm osc})}\,|P,2n\mp1;\pm\rangle\rangle$$

$$\text{with}\qquad \Psi^{\rm R,\pm}_{C,{\rm cont},J=\frac{1}{2}}(P,2n\mp1) = \frac{8}{\sqrt{2}}\,\frac{\Gamma(-2iP)\Gamma(1-2iP)}{\Gamma(\frac{1}{2}-iP+n)\Gamma(\frac{1}{2}-iP-n)}\,. \tag{2.17}$$

A similar exercise can be carried out for the $J=0$ crosscap state. The wavefunctions take the form

$$\begin{aligned}\Psi^{\rm NS,\pm}_{C,{\rm cont},J=0}(P,2n+1) &= \mp\frac{8}{\sqrt{2}}\,\cosh\pi P\,\frac{\Gamma(-2iP)\Gamma(1-2iP)}{\Gamma(1-iP+n)\Gamma(-iP-n)}\\ \Psi^{\rm R,\pm}_{C,{\rm cont},J=0}(P,2n) &= \mp\frac{8}{\sqrt{2}}\,e^{i\pi n}\,\cosh\pi P\,\frac{\Gamma(-2iP)\Gamma(1-2iP)}{\Gamma(1-iP+n)\Gamma(-iP-n)}\,.\end{aligned} \tag{2.18}$$

Thus, in the NSNS sector, we find that the $J=0$ crosscap has only odd winding modes while the RR sector wavefunctions have even winding modes.

A crosscap state that has a consistent projection in the open string channel on the identity character is the identity crosscap state, whose overlap with the D3 gives the twisted identity character. Its wavefunction is just the difference of the $J=0$ and $J=\frac{1}{2}$ crosscap states in equations (2.18) and (2.16). We will use this crosscap state to engineer our $SO/Sp$ gauge theory in the electric set-up. However, let us first complete the worldsheet analysis and use the crosscap wavefunctions to compute the Klein bottle amplitudes associated to the crosscap states with labels $J=1/2$ and $J=0$.

### 2.3 Klein bottle amplitudes

In order to interpret the crosscap states, it is useful to compute the Klein bottle amplitudes. When combined with the torus amplitude, the Klein bottle projects the bulk closed string



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spectrum. By computing the Klein bottle, we can identity the action of the orientifold projection operator on the closed strings.

Let us consider the $J = \frac{1}{2}$ case first. Using the explicit wavefunctions obtained in the previous section, we get, for the NS+NS+ overlap

$$\begin{aligned}\mathcal{K}^{(\frac{1}{2})}_{\rm NS+NS+} &= \frac{1}{2}\int \frac{ds}{128\pi^4}\frac{\vartheta_{00}(is)}{\eta^3(is)}\sum_{n\in\mathbb{Z}}\int_0^\infty dP\, 2^5\,\frac{\cosh 2\pi P+\cos 2n\pi}{(\sinh 2\pi P)^2}ch_c^{\rm NS}(p,n,is)\\ &= \frac{1}{2}\cdot 2^4\int \frac{ds}{128\pi^4}\frac{\vartheta_{00}(is)}{\eta^3(is)}\sum_{n\in\mathbb{Z}}\int_0^\infty dP\,\frac{q_c^{P^2+n^2}}{(\sinh\pi P)^2}\frac{\vartheta_{00}(is)}{\eta^3(is)}\\ &= \frac{1}{2}\cdot 2^4\int \frac{ds}{128\pi^4}\frac{\vartheta_{00}(is)}{\eta^3(is)}\int_0^\infty dP\,\frac{\vartheta_{00}(2is)}{(\sinh\pi P)^2}\,q_c^{P^2}\,\frac{\vartheta_{00}(is)}{\eta^3(is)}\,. \end{aligned} \tag{2.19}$$

Modular transforming to the $t$ channel using $s = \frac{1}{2t}$, we get

$$\begin{aligned}\mathcal{K}^{(\frac{1}{2})}_{\rm NS+NS+} &= \frac{1}{2}\int \frac{dt}{128\pi^4\,t^3}\frac{\vartheta_{00}(2it)}{\eta^3(2it)}\int_0^\infty\frac{dP}{(\sinh\pi P)^2}\,[\Theta_{0,1}(2it)+\Theta_{1,1}(2it)]\\ &\qquad\qquad\int_0^\infty dP'\,\cos 2\sqrt{2}\pi PP'\,q_o^{P'^2}\,\frac{\vartheta_{00}(2it)}{\eta^3(2it)}\\ &= \frac{1}{2}\int \frac{dt}{128\pi^4\,t^3}\frac{\vartheta_{00}(2it)}{\eta^3(2it)}\int_0^\infty dP'\,\big[\rho^{\rm NS}_{1,KB}(P'){\rm Ch}^{\rm NS}(P',0,2it)\\ &\qquad\qquad+\rho^{\rm NS}_{2,KB}(P'){\rm Ch}^{\rm NS}(P',1,2it)\big]\,. \end{aligned} \tag{2.20}$$

where we have used the identity

$$\vartheta_{00}(\tau) = \vartheta_{00}(4\tau)+\vartheta_{10}(4\tau)\,. \tag{2.21}$$

Here, we have defined the extended characters (at level 1) [11]

$${\rm Ch}^{\rm NS}(P,m,\tau) = q^{P^2}\,\Theta_{m,1}(\tau)\,\frac{\vartheta_{00}(\tau)}{\eta^3(\tau)}\,, \tag{2.22}$$

and the density of states

$$\rho^{\rm NS}_{1,KB}(P') = \rho^{\rm NS}_{2,KB}(P') = \int_0^\infty dP\frac{\cos 2\sqrt{2}\pi PP'}{(\sinh\pi P)^2}\,. \tag{2.23}$$

We note that the exchange of momentum integrals in $P$ and $P'$ is allowed in this case — there are no subtleties such as those discussed in [28].

One can do similar calculations for the NS+NS− and R+R+ as well. One can then check that the total Klein bottle amplitude vanishes. These calculations as well as the analogous ones for the $J = 0$ crosscap state are carried out in appendix C. One important difference between the $J = 0$ and the $J = \frac{1}{2}$ Klein-bottle computation is that, unlike the infrared divergent density of states in (2.23), the density of states for the $J = 0$ crosscap is a $\delta$-function at $P = 0$. This point will be crucial later on when we engineer the gauge theories of interest and we will come back to this point shortly, but we first give a closed string interpretation of these crosscap states in terms of projection operators.

**Interpretation**

- Let us first study the bulk superstring theory in which we only add the $J = 1/2$ crosscap state. From comparison of the asymptotics (or volume divergence) of the Klein-bottle amplitude and the torus amplitude, we conclude that the bulk orientifold is the one obtained from modding out the type IIB non-critical superstring theory by the operation $\Omega$. Indeed the Klein bottle amplitude (2.20) has the interpretation as a trace over the closed string Hilbert space with $\Omega$ inserted.

- If we consider the bulk theory supplemented with the crosscap state $J = 0$ only, then from the Klein-bottle amplitude, we derive that we have the type IIB bulk theory modded out by the operation $\Omega(-1)^n$. The operator $(-1)^n$ gives an extra minus sign to all Klein-bottle contributions with odd momentum $n$ along the angular direction of the cigar, compared to the Klein bottle amplitude with only $\Omega$ inserted. We note that the operation $(-1)^n$ is coded in coordinate space as the shift $\theta \to \theta + \pi$, which is a symmetry of the cigar winding condensate *only* on the condition that the parameter $\mu$ is *real*. Moreover, geometrically, the action only has a fixed point at the tip of the cigar, leading to the fact that the corresponding Klein bottle amplitude does not exhibit a volume divergence (in the dilaton direction).

- The $J = \frac{1}{2}$ crosscap has a continuous density of states (2.23) which diverges as $P \to 0$ (like the D5 branes) whereas the $J = 0$ crosscap, as shown in appendix C, has a $\delta$-function density localized strictly at $P = 0$. This tells us that the former is extended along the cigar direction while the latter is localized at the tip. We can also verify this from the form of the wavefunctions in equations (2.16), (2.18).

- Finally, when we add both crosscap states to the theory, we observe that they do not generate cross-terms in the Klein-bottle amplitude, due to the orthogonality of the Ishibashi states. When we add both Klein bottle amplitudes to the torus amplitude, they generate a combined $(1 + \Omega + \Omega(-1)^n)$ orientifold/orbifold. In a closed string theory modded out by the above two actions, one would be forced (from the closure of the group) to include an orbifold projection $(-1)^n$ as well, leading to a projection of the form $\frac{1}{4}(1 + \Omega)(1 + (-1)^n)$.

- We note that the $J = 1/2$ crosscap states only exchange massive strings, including massive gravitons. The exchanged states have worldsheet vertex operators that differ in conformal dimensions by integers. The crosscap state is not charged under massless RR fields.

- The $J = 0$ crosscap states exchanges massless closed strings, including the winding tachyon in the NSNS sector, and the massless RR scalar. Again the exchanged states differ by an integer in their worldsheet conformal dimensions.

We move on to finalize our analysis of amplitudes in the unoriented theory. We compute the overlap of the crosscap states with the flavour D5-branes.



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### 2.4 D5 brane Möbius amplitudes

We start out by calculating the overlap of the $J = \frac{1}{2}$ crosscap with the $J = \frac{1}{2}$ D5 brane in the NS+NS− sector as this corresponds to an NS sector open string amplitude.

$$M_\Omega^{\rm NS}\left(D5, J=\frac{1}{2}; J=\frac{1}{2}\right) = 2\int \frac{ds}{128\pi^4}\sum_{n\in\mathbb{Z}}\int_0^\infty dP \Psi^{{\rm NS},\mp}_{{\rm cont},J=\frac{1}{2}}(-P,2n)\Psi^{{\rm NS},\pm}_{C,{\rm cont},J=\frac{1}{2}}(P,2n) \\ q^{P^2+n^2}\left[\frac{e^{\frac{\pi i}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})}\right]^2 . \quad (2.24)$$

Substituting the wavefunctions using equations (2.6) and (2.16), one gets

$$M_\Omega^{\rm NS}\left(D5, J=\frac{1}{2}; J=\frac{1}{2}\right) = 4i\int \frac{ds}{128\pi^4}\sum_{n\in\mathbb{Z}}\int_0^\infty \frac{dP}{(\sinh\pi P)^2}\, e^{i\pi n}\, q_c^{P^2+n^2}\left[e^{\frac{i\pi}{8}}\frac{\vartheta_{01}(is+\frac{1}{2})}{\eta^3(is+\frac{1}{2})}\right]^2 \\ = 4i\int \frac{ds}{128\pi^4}\int_0^\infty \frac{dP}{(\sinh\pi P)^2} q_c^{P^2}\vartheta_{01}(2is)\left[e^{\frac{i\pi}{8}}\frac{\vartheta_{01}(is+\frac{1}{2})}{\eta^3(is+\frac{1}{2})}\right]^2 . \quad (2.25)$$

Modular transforming this to the open channel using $s = \frac{1}{4t}$, we obtain

$$M_\Omega^{\rm NS}\left(D5, J=\frac{1}{2}; J=\frac{1}{2}\right) = \int \frac{dt}{128\pi^4 t^3}\frac{e^{\frac{\pi i}{8}}\vartheta_{00}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})}\int_0^\infty dP'\,\rho^{\rm NS}_{2,M}(P')\,{\rm Ch}^{\rm NS}_\Omega(P',1,it)$$

$$\text{where} \qquad {\rm Ch}^{\rm NS}_\Omega(P,m,\tau) = q^{P^2}\,\Theta_{m,1}(\tau)\,\frac{e^{\frac{i\pi}{8}}\vartheta_{00}(\tau+\frac{1}{2})}{\eta^3(\tau+\frac{1}{2})} \quad (2.26)$$

$$\text{and} \qquad \rho^{NS}_{2,M} = \int_0^\infty dP\,\frac{\cos 2\pi PP'}{(\sinh\pi P)^2}\,.$$

Here we have used the identity $\vartheta_{10}(\tau+1) = e^{\frac{i\pi}{4}}\vartheta_{10}(\tau)$ and defined the $\Omega$-twisted extended character ${\rm Ch}_\Omega$. We observe that this modifies the density of states $\rho_2$ in the D5 self overlap (with a divergent contribution) [11, 7]. Note that the constants conspire so that in the $P \to 0$ limit of the sum of the cylinder and Möbius amplitudes, the projection in the open string channel is purely by $\Omega$ on the massless open string states. A similar computation yields the characters for the other sectors ${\rm Ch}^{\widetilde{\rm NS}}_\Omega(P',1,it)$ and ${\rm Ch}^{\rm R}_\Omega(P',1,it)$ with the same density of states (2.26).

Let us repeat this exercise for the overlap of the $J = 0$ crosscap with the $J = \frac{1}{2}$ D5-brane in the NS+NS− sector:

$$M_\Omega^{\rm NS}\left(D5, J=\frac{1}{2}; J=0\right) = -4\int \frac{ds}{128\pi^4}\int_0^\infty \frac{dP}{\cosh\pi P}\vartheta_{10}(2is)\, q_c^{P^2}\left[e^{\frac{i\pi}{8}}\frac{\vartheta_{01}(is+\frac{1}{2})}{\eta^3(is+\frac{1}{2})}\right]^2 . \quad (2.27)$$

Modular transforming this into the open string channel, we get

$$M_\Omega^{\rm NS}(D5, J=\frac{1}{2}; J=0) = \int \frac{dt}{128\pi^4 t^3}\frac{e^{\frac{\pi i}{8}}\vartheta_{00}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})}\int_0^\infty dP'\,\rho^{\rm NS}_{1,M}(P')\,{\rm Ch}^{\rm NS}_\Omega(P',0,it),$$

$$\rho^{\rm NS}_{1,M} = \int_0^\infty dP\,\frac{\cos 2\pi PP'}{(\cosh\pi P)}\,. \quad (2.28)$$

A similar computation yields the characters for the other sectors $\widetilde{\mathrm{Ch}}^{\widetilde{\mathrm{NS}}}_{\Omega}(P',0,it)$ and $\mathrm{Ch}^{\mathrm{R}}_{\Omega}(P',0,it)$ with the same density of states (2.28). As shown in appendix B, where all the amplitudes in the other channels have been computed, the total Möbius amplitude vanishes in each case:

$$M^{\mathrm{NS}}_{\Omega} - M^{\tilde{\mathrm{NS}}}_{\Omega} - M^{\mathrm{R}}_{\Omega} = 0 \tag{2.29}$$

We therefore observe that the boundary states and crosscap states that we considered are all mutually space-time supersymmetric. This concludes the calculation of amplitudes that will be relevant for the gauge theory to be constructed in the next section. These Möbius amplitudes will be instrumental in inferring what the global symmetry group is for the gauge theories we construct in the next section. For now, we conclude this section with the following observation:

- The $J = 0$ crosscap overlap with the $J = \frac{1}{2}$ extended brane gives a density of states which is spread a little around $P' = 0$ but decays rapidly as $P' \to 0$. This is consistent with the fact that the $J = 0$ crosscap is localized at the tip of the cigar. It is interesting that unlike the previous examples studied for localized D-branes, we do not get a sharp $\delta-$function in this case for the localized crosscap.

## 3. Microscopic description of Seiberg duality

### 3.1 The electric set-up

We now have all the ingredients, the boundary states and crosscap states, to engineer the gauge theories of interest to us. We start with the electric description of the gauge theory. To engineer an $\mathrm{SO}(N_c)$ gauge theory with $N_f$ flavours, we consider a configuration of $N_c$ $|D3\rangle$ branes, $N_f$ $|D5; J = \frac{1}{2}\rangle$ branes and an orientifold state $|C,\mu\rangle$ which is the difference of the $J = \frac{1}{2}$ and $J = 0$ crosscap states discussed in the earlier sections

$$|C,\mu\rangle = |C, J = M = 1/2, \mu\rangle - |C, J = M = 0, \mu\rangle. \tag{3.1}$$

We first discuss the pure gauge theory degrees of freedom.

#### 3.1.1 Pure gauge theory

The orientifold does not act on the gauge field vertex operators themselves, but has an action on the Chan-Paton factors. Effectively, it makes the $3-3$ strings unoriented. In the presence of the orientifold, the $D3$ branes realize a four-dimensional $\mathrm{SO}(N_c)$ gauge theory with $\mathcal{N} = 1$ supersymmetry on their worldvolume. To see this, one can combine the D3-brane cylinder amplitude in the oriented theory, divided by 2 (from the orientifold projection operator) and multiplied by $N_c$ squared, and the Möbius amplitudes of subsection 2.2, divided by 2, and multiplied by $N_c$ to obtain the amplitude that codes the open string spectrum projected by the orientifold operation. The first term in the expansion of the amplitude, for the $\mathrm{SO}(N_c)$ gauge theory is then:

$$Z = \frac{N_c^2 - N_c}{2}.(2-2) + \ldots \tag{3.2}$$

which has the interpretation as counting the transverse polarizations of a vector field in the adjoint of the $\mathrm{SO}(N_c)$ gauge group (as well as the corresponding gauginos). The $Sp$ gauge group is realized by taking the opposite (overall) sign of the orientifold plane.

We can confirm the above picture as follows. From the ratio of one-point functions for the RR scalar, we can read of the charge of the crosscap state relative to the charge of the D5-brane. A short calculation gives that the ratio of the $J = 0$ crosscap charge to that of the D5-brane is $-4$. It is therefore $+2$ times the charge of an unmirrored D3-brane. Therefore minus the $J = 0$ crosscap state projects onto $SO$ gauge theories on the D3-brane.

We note in passing that for the gauge theory without flavours, we can argue following [17] that the breaking of the anomalous $\mathrm{U}(1)_R$ is encoded in the backreaction of the D3-branes *and* the orientifold plane on the RR-scalar. The scalar will develop a dependence on the angular cigar variable that is proportional to $N_c \mp 2$ for $SO/Sp$ gauge theories. Taken into account the properly quantized shift symmetry of the RR-scalar, this then codes the breaking of $\mathrm{U}(1)_R$ to $Z_{2\check{h}}$ in the quantum theory (where $\check{h}$ is the dual coxeter number of the gauge algebra).

### 3.1.2 Adding flavour

We now turn to adding flavour. The $J = \frac{1}{2}$ brane does not have any massless localized modes in an overlap with either of the crosscaps. This may be seen from the small $q$ expansions of the Möbius overlaps in section 2.4 and in the appendix B. Thus the only other massless four-dimensional modes in this setup arise from the unoriented $D3 - D5$ strings. These are chiral superfields $Q^{ia}, a = 1, \ldots, N_c, i = 1, \ldots, N_f$ which fall into the vector representation of $\mathrm{SO}(N_c)$. The theory on the worldvolume of the branes at low energies is thus a four-dimensional $\mathrm{SO}(N_c)$ gauge theory with $N_f$ quarks, which has $\mathcal{N} = 1$ supersymmetry.

From the D5 Möbius amplitudes, we also read off that the global symmetry group is indeed $\mathrm{Sp}(N_f)$. This can be seen as follows: while the massless D3-brane spectrum is projected by both the $J = 0$ and the $J = 1/2$ crosscap, the D5-brane massless spectrum is only affected in the open string channel by the non-compact $J = 1/2$ crosscap. This can be checked by expanding the Möbius amplitudes for the D5-brane with the $J = 0$ crosscap and finding only massive modes in the open string channel (see equations (B.10), (B.11), (B.12)). Moreover, by expanding the Möbius amplitudes that project the massless modes in the open string channel for the D3-brane (see subsection B.1) compared to the D5-brane (see subsection B.2), we see that the Möbius amplitudes have a relative minus sign. This can be traced back to the relative sign in the full orientifold that corresponds to subtraction of the $J = 0$ and the $J = 1/2$ crosscaps (plus the fact that the main contribution to the D3-brane Möbius amplitude is from the $J = 0$ crosscap). The relative minus sign between the D3 and D5 Möbius amplitudes codes the fact that the global symmetry group is $Sp$ when the gauge group is $SO$ (and vice versa). We believe this to be a relevant check on our brane set-up.

## 3.2 Worldsheet analysis of $\mu-$transitions and the magnetic theory

Now that we have identified the relevant boundary states and crosscaps, we can proceed,



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following [7], to infer Seiberg duality from the worldsheet. In what follows, we use the known transformations of the D-brane states under the $\mu$-flip transformation $\mu \to -\mu$, the consequent rearrangement of the D-branes in the charge lattice, mutual supersymmetry with the final $\mu$-flipped configuration as well as some physical statements about tensions to infer the behaviour of the crosscap states under the $\mu-$flip. We first review the transformation of the boundary states following. Since most of the details can be found in [7], we will be brief in our discussion.

### 3.2.1 Behaviour of boundary states and crosscaps under $\mu-$transitions

The behavior of the D3 and D5-brane boundary states under a sign-flip of the coefficient $\mu$ of the cigar interaction term (i.e. the N=2 Liouville interaction term corresponding to a winding condensate) was determined in [7]. The first ingredient in the derivation is the fact that the bulk vertex operators satisfy a reflection relation, and therefore their one-point functions satisfy it as well [9]. This determines the $\mu$ dependence of the one-point functions for the boundary states, as well as the crosscap states. The $\mu$-dependence of the closed string one-point functions in the NSNS sector amounts to an overall factor of $\mu^{iP-Q/2}\overline{\mu}^{iP+Q/2}$ where $P$ is the momentum of the closed string, and $Q$ is the (worldsheet left-moving) $\mathrm{U}(1)_R$ charge of the closed string vertex operator.

We recall that D3-branes are localized objects with a well-defined mass. The gravitational backreaction is dominantly on the on-shell mode with asymptotic behaviour $e^{-2\rho}$, $\rho$ being the radial direction of the cigar [17]. Under $\mu \to -\mu$, the one-point function for this mode flips sign. That shows that the NSNS part of the D3-brane boundary state needs to pick up an explicit minus sign, when we demand that it maps into a physical D-brane with positive mass [7].

For the NSNS part of the D5-brane boundary state, the dominant backreaction is onto the cosmological constant, which does not pick up a minus sign under the $\mu \to -\mu$ transformation. Moreover, since the D5-brane changes orientation under the $\mathbb{Z}_2$ transformation [7], the RR part of the boundary state flips sign. Charge conservation then fixes that the D3-brane RR boundary state flips sign as well (leading to a simple overall minus sign in the boundary state).

Thus, under the $\mathbb{Z}_2$ transformation, we have the following transformation of the boundary states [7]:

$$\begin{aligned} N_c\,|D3,\mu\rangle &\longrightarrow N_c\,(-|D3,-\mu\rangle) \\ N_f\,\overline{|D5,J=1/2=M,\mu\rangle} &\longrightarrow N_f\,|D5,J=1/2=M,-\mu\rangle \\ &\longrightarrow N_f\,\overline{|D5,J=M=0,-\mu\rangle} + N_f\,(-\overline{|D3,-\mu\rangle})\,. \end{aligned} \tag{3.3}$$

After the annihilation of the $N_c$ D3-branes with $N_f$ anti-D3-branes (for $N_f \geq N_c - 2$), we remain with $N_f - N_c$ color anti-branes, as well as $N_f$ mutually supersymmetric flavor branes (at $J = 0$). Thus, on the magnetic side, the colour gauge theory is realized on $N_f - N_c$ branes of the form $(-\overline{|D3,-\mu\rangle})$.

We now turn to the determination of the behaviour of the crosscap states under the $\mathbb{Z}_2$ transformation. Our attitude will be slightly different than in the D-brane case. We

will only attempt to describe the final configuration that results after the system has fully relaxed to the supersymmetric configuration. To that end, we would like to understand the crosscaps which are supersymmetric with respect to the final configuration of D-branes above in the magnetic theory at $-\mu$. Mutual supersymmetry with the branes $(-\overline{|D3, -\mu\rangle})$ and $\overline{|D5, J = M = 0, -\mu\rangle}$ dictates that the relative sign of the NSNS and RR parts of the crosscap states change (as for the D-branes) when we go from $\mu \to -\mu$. Moreoever from the pure bulk theory before double scaling, we know that the exchange of the NS5 and NS5' branes before any D-branes are present is a trivial operation. That remains true in the presence of the orientifold plane, so the Klein-bottle amplitude does not change under the $\mu$-flip.

We further expect that the gauge theory which is realized on the localized branes is projected onto an SO group as before. This projection arises from a Möbius amplitude which is an overlap of the new crosscap with the localized branes in the magnetic theory $(-\overline{|D3, -\mu\rangle})$. These conditions fix the transformation of the crosscap:

$$|C, \mu\rangle \longrightarrow -\overline{|C, \mu\rangle}\,. \tag{3.4}$$

We can further argue for this transformation rule by observing that the tension of the $J = 0$ crosscap is measured by its coupling to the tachyon winding mode and is proportional to $\mu$. Like the localized D-brane, this also changes sign with $\mu$. That leads to the explicit minus sign in the transformation rule.[3]

The charge of the final state is opposite to the initial state when measured correctly with the new axion which has also changed sign. Unlike the D-brane case, we cannot argue at the level of boundary/crosscap states that there is a rearrangment of the basis in the final state, since charge is actually not conserved in this situation described above. The only possibility consistent with charge conservation, as we shall consider below, is that there are new charge-carrying localized states which are created during the transition. These will turn out to be D3 branes.

Intuitively speaking, the localized part of the orientifold, namely the $J = 0$ crosscap, accounts for the charge difference between the orientifold four-planes that end on a given NS5-brane from opposite direction in the type IIA set-up. We will see this more precisely in the next section where we exhibit pairs of Seiberg dual gauge theories. It will turn out that the axionic charge measured by the RR scalar in the worldsheet description keeps track of the spacetime linking number in the IIA set-up.

Finally, we note that the $\mathbb{Z}_2$ flip on the parameter $\mu$ can be generalized to a transformation of the parameter $\mu$ along the *real line*. (The parameter $\mu$ needs to be real in order for the $\Omega(-1)^n$ orientifold to exist. Another way to see this is to observe that there is no Fayet-Iliopoulos parameter to turn on for the $SO/Sp$ gauge theories.) Thus, in contrast to the non-critical set-up for $SU$ gauge theories, we necessarily go through a region of strong

[3]We should note that it is not obvious in this formalism that the tension of the magnetic orientifold plane in the asymptotic region, as measured in the overlap with the magnetic flavour brane remains positive (as appropriate for an orientifold plane that projects onto an $Sp$ global symmetry group). This is hidden in the fact that the propagator appearing in the calculation of the overlap is $\mu$ dependent.



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coupling. This is the basic reason why we here take the attitude to describe the final orientifold configuration from scratch, without attempting to follow it closely through the whole transition process. As shown in [19], it is this motion through the strong coupling region that creates of the extra localized charged D-brane states, which leads to a conservation of the axionic charge in the process.

### 3.2.2 Seiberg duality

It will be useful to be explicit about how charge is conserved under the $\mathbb{Z}_2$ transformation. First of all, in the initial configuration, we measure axionic charge with respect to the massless RR scalar. In the magnetic theory, we will measure the axionic charge with respect to a sign-flipped RR scalar (since it flips sign under the $\mathbb{Z}_2$ transformation). In the electric theory we have $N_c$ (unmirrored) color D3-branes and $N_f$ flavor anti-D5 (J=1/2) branes, giving rise to an $\mathrm{SO}(N_c)$ gauge theory with $\mathrm{Sp}(N_f)$ flavour group.

Recalling that in the oriented theory, the axionic charge of the D5 is $\frac{1}{2}$ that of the D3 [7], this configuration contributes to the axionic charge $N_c - \frac{1}{2}N_f$. From the previous section, we conclude that the orientifold plane contributes $-2$ to the charge, for a total axionic charge of $N_c - \frac{1}{2}N_f - 2$. As mentioned earlier, we observe that this coincides with the linking number $l_{NS}$ defined for the NS brane in the type IIA set-up in (1.2). The charge also measures the anomaly in a specific linear combination of the axial flavour $\mathrm{U}(1)_A$ charge and the $\mathrm{U}(1)_R$ charge (which is proportional to $Q_R + 1/2Q_A$ in units where the R-charge of the gaugino and the axial charge of a left-handed quark are both equal to one).

In the magnetic description, after the annihilation of the $N_c$ D3-branes with $N_f$ anti-D3-branes (for $N_f \geq N_c - 2$), we remain with $N_f - N_c$ color anti-branes, as well as $N_f$ mutually supersymmetric $J = 0$ flavor branes. Furthermore we have a crosscap state $-\overline{|C;-\mu\rangle}$ which gives a total charge of $N_c - \frac{1}{2}N_f + 2$.

The axionic charge in the electric and magnetic configurations differ by four (in the units of $D3$ brane charge). Charge conservation therefore necessarily implies that four $D3$ branes must be created while passing through the strong-coupling region at $\mu = 0$, just as in the discussion of Seiberg duality in the type IIA NS5 brane set up [4]. Thus the gauge group in the magnetic theory is $\mathrm{SO}(N_f - N_c + 4)$. As in the electric setup, there are no new massless modes being created by the orientifold. Similarly, for the same reasons as in the electric set-up, the global symmetry group is $\mathrm{Sp}(N_f)$ also in the magnetic configuration.

The other massless localized modes in the magnetic setup are the magnetic quarks arising from the $D3 - D5$ strings and the meson from the $D5 - D5$ overlap. As in the $\mathrm{SU}(N)$ case [7], the overlap of the extended brane at $J = 0$ gives rise, in addition to a six-dimensional continuum of modes, to a *four dimensional localized* massless scalar with the quantum numbers of the meson. This arises from a correct convergent definition of the overlap which involves a contour prescription [28, 7].[4]

[4]This is the only mode that arises in the localized contribution of the $D5$ overlap. There is another massless localized mode with the quantum numbers $A^{ia}$ which arises in the $D3 - D5(J = 0)$ overlap, but the symmetries forbid a minimal gauge coupling of this mode with any of the massless fields. This is interpreted as the fact that this gauge field gets a mass, as suggested by the ten-dimensional picture [3].



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To summarize, the magnetic theory has gauge fields, quarks, as well as mesonic degrees of freedom. As in the electric setup, the strings fluctuations which gave rise to the quarks and mesons are now also unoriented. The matter content of this gauge theory is $N_f$ quarks $q^{ia}, a = 1, \ldots, (N_f - N_c), i = 1, \ldots, N_f$ in the vector representation and a meson $M^{ij}$ in the symmetric representation of the $\mathrm{Sp}(N_f)$ global flavor symmetry.

## 4. Conclusions

The main technical result in our work is the construction of exact crosscap states in the cigar background using the modular bootstrap method. Using these states in conjunction with the known D-brane boundary states [16, 17] in this background, we engineered four dimensional $SO/Sp$ gauge theories with fundamental flavours. We found that microscopically, we could interpret Seiberg duality as a re-arrangement of the basis of boundary states and crosscaps under the $\mu \to -\mu$ transformation in the conformal field theory moduli space.

The exact microscopic realization of Seiberg duality in $\mathcal{N} = 1$ $SO/Sp$ gauge theories with flavour is rendered more subtle than its $\mathrm{SU}(N_c)$ counterpart [7] by the fact that the $\mu-$transition necessarily takes us through the strong coupling region. In the exact realization this is expressed through the fact that extra boundary states need to be created while going through the strong coupling region of the gauge theory. This was derived in the non-critical set-up by using a charge conservation argument similar to the one used in the context of the IIA brane set-ups we discussed in the introduction. The relevant charge in this case turns out to be the axionic charge measured by the RR scalar.

The crosscap states that we used to engineer the gauge theory with flavour still have a geometric interpretation in this highly curved regime of string theory. In principle, one would like to connect these orientifold crosscap states to their asymptotically flat ancestors, tracking them through the double scaling limit [6], thereby connecting them to geometric orientifold planes in flat space. One can use the added control over these states afforded by the CFT to probe in detail the gauge theory on the branes beyond the information which can be read off from the singular ten-dimensional setup. As discussed in [7], the full theory on the branes at non-zero energies could have contributions from other open and closed string modes. Regarding this point, we note that in the presence of the orientifolds we must also take into account any potential contributions from the closed string twisted sectors arising from the various Klein bottles.

We only discussed boundary states and crosscaps based on the continuous representations of the $\mathcal{N} = 2$ superconformal algebra. These are double-sheeted objects on the cigar and as a result, there were no constraints coming from tadpole cancellation. However, it is probable that including boundary states and crosscaps based on discrete representations would allow us to include chiral matter into the gauge theory through clever cancellation of RR tadpoles arising from D-branes and orientifold planes. We leave this as an open problem.

Finally, it would be very interesting to generalize the implementation of Seiberg duality in non-critical strings to more general gauge theories with product gauge groups and bi-fundamental matter obtained by putting supersymmetric branes in (oriented or unoriented)



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non-compact Gepner models. Such theories have been studied in the context of geometric engineering in [29]. It would be interesting to see if the description of Seiberg duality given in that reference (for instance, as Weyl reflections of the simple roots in the special case of the ADE quivers) can be given a worldsheet description. When the moduli of the conformal field theory are complex, we expect Seiberg duality to emerge as a monodromy in the space of couplings. When orientifolds are present, based on the simple example studied here, we expect that the duality will manifest itself as a rearrangement of boundary states and crosscaps under discrete actions on the space of couplings of suitable Gepner (boundary) conformal field theories.


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## A. Conventions

Our definitions for the $\vartheta_{ab}$ functions are:

$$\begin{aligned}
\vartheta_{00}(\tau,\nu) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}\, z^n \\
\vartheta_{01}(\tau,\nu) &= \sum_{n=-\infty}^{\infty} (-1)^n\, q^{n^2/2}\, z^n \\
\vartheta_{10}(\tau,\nu) &= \sum_{n=-\infty}^{\infty} q^{(n-1)^2/2} z^{n-1/2} \\
\vartheta_{11}(\tau,\nu) &= -i \sum_{n=-\infty}^{\infty} (-1)^n\, q^{(n-1)^2/2} z^{n-1/2}\,,
\end{aligned} \tag{A.1}$$

where $q = e^{2\pi i\tau}$ and $z = e^{2\pi i\nu}$. We also define the level $k$ $\Theta$ functions:

$$\Theta_{m,k}(\tau,\nu) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2}\, z^{k(n+\frac{m}{2k})}\,. \tag{A.2}$$

We will only use the level $k=1$ $\Theta_{m,1}(\tau,\nu)$ functions with $m = 0, 1$. The level 1 $\Theta_{m,1}(\tau,\nu)$ functions are related to the $\vartheta(\tau,\nu)$ functions as follows:

$$\Theta_{0,1}(\tau,\nu) = \vartheta_{00}(2\tau,\nu) \qquad \Theta_{1,1}(\tau,\nu) = \vartheta_{10}(2\tau,\nu)\,. \tag{A.3}$$

The following identities are handy in verifying space-time supersymmetry:

$$\begin{aligned}\vartheta_{00}^2(\tau) &= \vartheta_{00}^2(2\tau) + \vartheta_{10}^2(2\tau)\\ \vartheta_{01}^2(\tau) &= \vartheta_{00}^2(2\tau) - \vartheta_{10}^2(2\tau)\\ \vartheta_{10}^2(\tau) &= 2\,\vartheta_{00}(2\tau)\vartheta_{10}(2\tau)\,.\end{aligned} \tag{A.4}$$

Apart from the usual modular transformation rules of the $\vartheta$-functions, the following particular modular transformations are useful in the unoriented theory [15]:

$$\begin{aligned}\vartheta_{00}\Big(\frac{i}{4s}+\frac{1}{2}\Big) &= \sqrt{2s}\,e^{\frac{i\pi}{4}}\,\vartheta_{01}\Big(is+\frac{1}{2}\Big)\\ \vartheta_{01}\Big(\frac{i}{4s}+\frac{1}{2}\Big) &= \sqrt{2s}\,e^{-\frac{i\pi}{4}}\,\vartheta_{00}\Big(is+\frac{1}{2}\Big)\\ \vartheta_{10}\Big(\frac{i}{4s}+\frac{1}{2}\Big) &= \sqrt{2s}\,\vartheta_{10}\Big(is+\frac{1}{2}\Big)\\ \eta\Big(\frac{i}{4s}+\frac{1}{2}\Big) &= \sqrt{2s}\,\eta\Big(is+\frac{1}{2}\Big)\,.\end{aligned} \tag{A.5}$$

## B. Modular transformation of Möbius amplitudes

### B.1 Overlaps of crosscaps with D3 branes

Let us consider equation (2.7) and modular transform the expression into the closed string channel. Performing the momentum integration, we get

$$\int \frac{d^4k}{16\pi^4} e^{2\pi i\tau k^2} = -\frac{1}{16\pi^4}\,\frac{1}{4\tau^2} = \frac{s^2}{4\pi^2}\,. \tag{B.1}$$

Let us consider the first term in equation (2.7): using the identities in (A.4) the integral over the modular parameter can be written as

$$M_\Omega^{\rm NS}\Big(D3; J=\frac{1}{2}\Big) = \int ds \frac{s}{8\pi^4}\,\frac{\vartheta_{00}(\frac{i}{4s}+\frac{1}{2})}{\eta(\frac{i}{4s}+\frac{1}{2})^3}\,e^{\frac{i\pi}{4}}\,\vartheta_{10}(\frac{i}{2s})\,\frac{\vartheta_{00}(\frac{i}{4s}+\frac{1}{2})}{\eta(\frac{i}{4s}+\frac{1}{2})^3}\,. \tag{B.2}$$

Using the modular transformations (A.5) this can be written in the closed string channel as

$$\begin{aligned}M_\Omega^{\rm NS}\Big(D3; J=\frac{1}{2}\Big) &= e^{\frac{i\pi}{4}}\int ds\,\frac{s}{8\pi^4}\left[\frac{1}{2s}\,\frac{e^{\frac{i\pi}{4}}\,\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\right]\left[\frac{1}{\sqrt{2s}}\frac{e^{\frac{i\pi}{4}}\,\vartheta_{01}(is+\frac{1}{2})\vartheta_{01}(2is)}{\eta(is+\frac{1}{2})^3}\right]\\ &= e^{\frac{i\pi}{2}}\int \frac{ds}{16\sqrt{2s}\,\pi^4}\,\frac{e^{\frac{i\pi}{8}}\,\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\,\vartheta_{01}(2is)\,\frac{e^{\frac{i\pi}{8}}\,\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\\ &= 8e^{\frac{i\pi}{2}}\int \frac{ds}{128\pi^4}\,\frac{e^{\frac{i\pi}{8}}\,\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\int_0^\infty dP\,2\,\vartheta_{01}(2is)\,q_c^{P^2}\,\frac{e^{\frac{i\pi}{8}}\,\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\,.\end{aligned} \tag{B.3}$$

where $q_c = e^{-2\pi s}$ as usual. Similar manipulations for the $\tilde{\text{NS}}$ and R sector terms in (2.4) yield, after modular transformation,

$$M_\Omega^{\tilde{\text{NS}}}\left(D3; J=\frac{1}{2}\right) = -8e^{-\frac{i\pi}{2}}\int \frac{ds}{128\pi^4}\,\frac{e^{\frac{i\pi}{8}}\vartheta_{00}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\int_0^\infty dP\,2\,\vartheta_{01}(2is)\,q_c^{P^2}\,\frac{e^{\frac{i\pi}{8}}\vartheta_{00}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}$$
$$M_\Omega^{\text{R}}\left(D3; J=\frac{1}{2}\right) = 8\int \frac{ds}{128\pi^4}\,\frac{\vartheta_{10}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\int_0^\infty dP\,2\,\vartheta_{10}(2is)\,q_c^{P^2}\,\frac{\vartheta_{10}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\,. \tag{B.4}$$

The $J=0$ Möbius amplitudes can be similarly modular transformed as follows:

$$M_\Omega^{\text{NS}}(D3; J=0) = 8\int \frac{ds}{128\pi^4}\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\int_0^\infty dP 2\cosh\pi P\vartheta_{10}(2is)\,q_c^{P^2}\frac{e^{\frac{i\pi}{8}}\vartheta_{01}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}$$
$$M_\Omega^{\tilde{\text{NS}}}(D3; J=0) = -8\int \frac{ds}{128\pi^4}\,\frac{e^{\frac{i\pi}{8}}\vartheta_{00}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\int_0^\infty dP 2\cosh\pi P\vartheta_{10}(2is)q_c^{P^2}\frac{e^{\frac{i\pi}{8}}\vartheta_{00}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}$$
$$M_\Omega^{\text{R}}(D3; J=0) = 8\int \frac{ds}{128\pi^4}\,\frac{\vartheta_{10}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\int_0^\infty dP\,2\cosh\pi P\,\vartheta_{01}(2is)\,q_c^{P^2}\,\frac{\vartheta_{10}(is+\frac{1}{2})}{\eta(is+\frac{1}{2})^3}\,. \tag{B.5}$$

One can check that the total Möbius amplitude vanishes:

$$M_\Omega^{\text{NS}} - M_\Omega^{\widetilde{\text{NS}}} - M_\Omega^{\text{R}} = 0 \tag{B.6}$$

### B.2 Overlaps of crosscaps with D5 branes

In this section, we merely record the overlaps of the $J=\frac{1}{2}$ and $J=0$ crosscap states with the $J=\frac{1}{2}$ and $J=0$ D-branes in the open string channel. These turn to be important to infer the flavour gauge group in the unoriented theory.

**Overlaps of $J=\frac{1}{2}$ D5 brane with $J=\frac{1}{2}$ crosscap.**

$$M_\Omega^{\text{NS}} = \int \frac{dt}{128\pi^4 t^3}\int_0^\infty \frac{dP}{(\sinh\pi P)^2}\int_0^\infty dP'\,\cos 2\pi PP' q_o^{P'^2}\Theta_{1,1}\left(it+\frac{1}{2}\right)\left[\frac{\vartheta_{00}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})}\right]^2. \tag{B.7}$$

$$M_\Omega^{\tilde{\text{NS}}} = -\int \frac{dt}{128\pi^4 t^3}\int_0^\infty \frac{dP}{(\sinh\pi P)^2}\int_0^\infty dP'\cos 2\pi PP' q_o^{P'^2}\Theta_{1,1}\left(it+\frac{1}{2}\right)\left[\frac{\vartheta_{01}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})}\right]^2. \tag{B.8}$$

$$M_\Omega^{\text{R}} = \int \frac{dt}{128\pi^4 t^3}\int_0^\infty \frac{dP}{(\sinh\pi P)^2}\int_0^\infty dP'\,\cos 2\pi PP' q_o^{P'^2}\Theta_{0,1}\left(it+\frac{1}{2}\right)\left[\frac{\vartheta_{10}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})}\right]^2. \tag{B.9}$$

**Overlaps of $J = \frac{1}{2}$ D5 brane with $J = 0$ crosscap.**

$$M_{\Omega}^{\rm NS} = e^{\frac{i\pi}{2}} \int \frac{dt}{128\pi^4 t^3} \int_0^\infty \frac{dP}{\cosh \pi P} \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{0,1}\left(it + \frac{1}{2}\right) \left[ e^{\frac{i\pi}{8}} \frac{\vartheta_{00}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.10}$$

$$M_{\Omega}^{\tilde{\rm NS}} = e^{\frac{i\pi}{2}} \int \frac{dt}{128\pi^4 t^3} \int_0^\infty \frac{dP}{\cosh \pi P} \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{0,1}\left(it + \frac{1}{2}\right) \left[ e^{\frac{i\pi}{8}} \frac{\vartheta_{01}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.11}$$

$$M_{\Omega}^{\rm R} = -e^{-\frac{i\pi}{4}} \int \frac{dt}{128\pi^4 t^3} \int_0^\infty \frac{dP}{\cosh \pi P} \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{1,1}\left(it + \frac{1}{2}\right) \left[ \frac{\vartheta_{10}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.12}$$

**Overlaps of $J = 0$ D5 brane with $J = \frac{1}{2}$ crosscap.**

$$M_{\Omega}^{\rm NS} = \int \frac{dt}{128\pi^4 t^3} \int_0^\infty \frac{\cosh \pi P dP}{(\sinh \pi P)^2} \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{1,1}\left(it + \frac{1}{2}\right) \left[ \frac{\vartheta_{00}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.13}$$

$$M_{\Omega}^{\tilde{\rm NS}} = -\int \frac{dt}{128\pi^4 t^3} \int_0^\infty \frac{\cosh \pi P dP}{(\sinh \pi P)^2} \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{1,1}\left(it + \frac{1}{2}\right) \left[ \frac{\vartheta_{01}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.14}$$

$$M_{\Omega}^{\rm R} = \int \frac{dt}{128\pi^4 t^3} \int_0^\infty \frac{\cosh \pi P dP}{(\sinh \pi P)^2} \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{0,1}\left(it + \frac{1}{2}\right) \left[ \frac{\vartheta_{10}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.15}$$

**Overlaps of $J = 0$ D5 brane with $J = 0$ crosscap.**

$$M_{\Omega}^{\rm NS} = -e^{\frac{i\pi}{2}} \int \frac{dt}{128\pi^4 t^3} \int_0^\infty dP \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{0,1}\left(it + \frac{1}{2}\right) \left[ e^{\frac{i\pi}{8}} \frac{\vartheta_{00}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.16}$$

$$M_{\Omega}^{\tilde{\rm NS}} = -e^{\frac{i\pi}{2}} \int \frac{dt}{128\pi^4 t^3} \int_0^\infty dP \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{0,1}\left(it + \frac{1}{2}\right) \left[ e^{\frac{i\pi}{8}} \frac{\vartheta_{01}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.17}$$

$$M_{\Omega}^{\rm R} = e^{-\frac{i\pi}{4}} \int \frac{dt}{128\pi^4 t^3} \int_0^\infty dP \int_0^\infty dP' \cos 2\pi P P' q_o^{P'^2} \Theta_{1,1}\left(it + \frac{1}{2}\right) \left[ \frac{\vartheta_{10}(it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})} \right]^2 . \tag{B.18}$$

In both cases, one can check that

$$M_{\Omega}^{\rm NS} - M_{\Omega}^{\widetilde{\rm NS}} - M_{\Omega}^{\rm R} = 0 . \tag{B.19}$$

## C. Klein bottle amplitudes

In this appendix, we collect some technical details about the Klein botle amplitudes. The computations of the NS+NS− and R+R+ contributions for the $J = \frac{1}{2}$ crosscap state leads to the expressions:

$$\begin{aligned}\mathcal{K}^{(\frac{1}{2})}_{\rm NS+NS-} &= \frac{1}{2}\,2^4 \int \frac{ds}{128\pi^4}\,\frac{\vartheta_{01}(is)}{\eta^3(is)} \int_0^\infty dP\,\frac{\vartheta_{00}(2is)}{(\sinh\pi P)^2}\,q_c^{P^2}\,\frac{\vartheta_{01}(is)}{\eta^3(is)} \\ &= \frac{1}{2}\int \frac{dt}{128\pi^4\,t^3}\,\frac{\vartheta_{10}(2it)}{\eta^3(2it)} \int_0^\infty \frac{dP}{(\sinh\pi P)^2}\,\left[\Theta_{0,1}(2it)+\Theta_{1,1}(2it)\right] \\ &\qquad\qquad \times \int_0^\infty dP'\,\cos 2\sqrt{2}\pi PP'\,q_o^{P'^2}\,\frac{\vartheta_{10}(2it)}{\eta^3(2it)}\,. \end{aligned} \tag{C.1}$$

$$\begin{aligned}\mathcal{K}^{(\frac{1}{2})}_{\rm R+R+} &= \frac{1}{2}\,2^4 \int \frac{ds}{128\pi^4}\,\frac{\vartheta_{10}(is)}{\eta^3(is)} \int_0^\infty dP\,\frac{\vartheta_{10}(2is)}{(\sinh\pi P)^2}\,q_c^{P^2}\,\frac{\vartheta_{10}(is)}{\eta^3(is)} \\ &= \frac{1}{2}\int \frac{dt}{128\pi^4\,t^3}\,\frac{\vartheta_{01}(2it)}{\eta^3(2it)} \int_0^\infty \frac{dP}{(\sinh\pi P)^2}\,\left[\Theta_{0,1}(2it)-\Theta_{1,1}(2it)\right] \\ &\qquad\qquad \times \int_0^\infty dP'\,\cos 2\sqrt{2}\pi PP'\,q_o^{P'^2}\,\frac{\vartheta_{01}(2it)}{\eta^3(2it)}\,. \end{aligned} \tag{C.2}$$

where we used the identity

$$\vartheta_{01}(\tau) = \vartheta_{00}(4\tau) - \vartheta_{10}(4\tau)\,. \tag{C.3}$$

All contributions to the crosscap self overlap are singular as $P \to 0$ and the total contribution to the Klein-bottle amplitude vanishes because of the identity

$$\vartheta_{00}(2is)\left(\frac{\vartheta_{00}^2(is)}{\eta^6(is)} - \frac{\vartheta_{01}^2(is)}{\eta^6(is)}\right) - \vartheta_{10}(2is)\frac{\vartheta_{10}^2(is)}{\eta^6(is)} = 0\,. \tag{C.4}$$

The calculations for the $J = 0$ crosscap are similar to the ones done above. We get:

$$\begin{aligned}\mathcal{K}^{(0)}_{\rm NS+NS+} &= 2^4 \int \frac{ds}{128\pi^4}\,\frac{\vartheta_{00}(is)}{\eta^3(is)} \int_0^\infty dP\,\vartheta_{10}(2is)\,q_c^{P^2}\,\frac{\vartheta_{00}(is)}{\eta^3(is)} \\ &= \int \frac{dt}{32\sqrt{2}\pi^4\,t^3}\,\frac{\vartheta_{00}(2it)}{\eta^3(2it)}\,\left[\Theta_{0,1}(2it)-\Theta_{1,1}(2it)\right]\,\frac{\vartheta_{00}(2it)}{\eta^3(2it)}\,. \end{aligned} \tag{C.5}$$

$$\begin{aligned}\mathcal{K}^{(0)}_{\rm NS+NS-} &= 2^4 \int \frac{ds}{128\pi^4}\,\frac{\vartheta_{01}(is)}{\eta^3(is)} \int_0^\infty dP\,\vartheta_{10}(2is)\,q_c^{P^2}\,\frac{\vartheta_{01}(is)}{\eta^3(is)} \\ &= \int \frac{dt}{32\sqrt{2}\pi^4\,t^3}\,\frac{\vartheta_{10}(2it)}{\eta^3(2it)}\,\left[\Theta_{0,1}(2it)-\Theta_{1,1}(2it)\right]\,\frac{\vartheta_{10}(2it)}{\eta^3(2it)}\,. \end{aligned} \tag{C.6}$$

$$\begin{aligned}\mathcal{K}^{(0)}_{\rm R+R+} &= 2^4 \int \frac{ds}{128\pi^4}\,\frac{\vartheta_{10}(is)}{\eta^3(is)} \int_0^\infty dP\,\vartheta_{00}(2is)\,q_c^{P^2}\,\frac{\vartheta_{10}(is)}{\eta^3(is)} \\ &= \int \frac{dt}{32\sqrt{2}\pi^4\,t^3}\,\frac{\vartheta_{01}(2it)}{\eta^3(2it)}\,\left[\Theta_{0,1}(2it)+\Theta_{1,1}(2it)\right]\,\frac{\vartheta_{01}(2it)}{\eta^3(2it)}\,. \end{aligned} \tag{C.7}$$

Note that in the $t$-channel the density of states is a delta function as a result of which there is no integral over the momenta in the radial direction of the cigar. Thus, the self overlap is regular as $P \to 0$ and vanishes because of the identity (in the $s$-channel)

$$\vartheta_{10}(2is)\left(\frac{\vartheta_{00}^2(is)}{\eta^6(is)}+\frac{\vartheta_{01}^2(is)}{\eta^6(is)}\right)-\vartheta_{00}(2is)\frac{\vartheta_{10}^2(is)}{\eta^6(is)}=0\,. \tag{C.8}$$

## References


[1] N. Seiberg, *Electric - magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[2] K. Intriligator, N. Seiberg and D. Shih, *Dynamical SUSY breaking in meta-stable vacua*, *JHEP* **04** (2006) 021 [hep-th/0602239].

[3] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[4] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici and A. Schwimmer, *Brane dynamics and $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 505** (1997) 202 [hep-th/9704104].

[5] A. Giveon, D. Kutasov and O. Pelc, *Holography for non-critical superstrings*, *JHEP* **10** (1999) 035 [hep-th/9907178].

[6] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *JHEP* **10** (1999) 034 [hep-th/9909110].

[7] S. Murthy and J. Troost, *D-branes in non-critical superstrings and duality in $N = 1$ gauge theories with flavor*, *JHEP* **10** (2006) 019 [hep-th/0606203].

[8] K.A. Intriligator and N. Seiberg, *Duality, monopoles, dyons, confinement and oblique confinement in supersymmetric* $\mathrm{SO}(N_c)$ *gauge theories*, *Nucl. Phys.* **B 444** (1995) 125 [hep-th/9503179].

[9] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory. I: boundary state and boundary two-point function*, hep-th/0001012;

[10] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[11] T. Eguchi and Y. Sugawara, *Modular bootstrap for boundary $N = 2$ Liouville theory*, *JHEP* **01** (2004) 025 [hep-th/0311141].

[12] C. Ahn, C. Rim and M. Stanishkov, *Exact one-point function of $N = 1$ super-Liouville theory with boundary*, *Nucl. Phys.* **B 636** (2002) 497 [hep-th/0202043].

[13] C. Ahn, M. Stanishkov and M. Yamamoto, *One-point functions of $N = 2$ super-Liouville theory with boundary*, *Nucl. Phys.* **B 683** (2004) 177 [hep-th/0311169].

[14] D. Israel, A. Pakman and J. Troost, *D-branes in $N = 2$ Liouville theory and its mirror*, *Nucl. Phys.* **B 710** (2005) 529 [hep-th/0405259].

[15] Y. Nakayama, *Crosscap states in $N = 2$ Liouville theory*, *Nucl. Phys.* **B 708** (2005) 345 [hep-th/0409039].

[16] A. Fotopoulos, V. Niarchos and N. Prezas, *D-branes and SQCD in non-critical superstring theory*, *JHEP* **10** (2005) 081 [hep-th/0504010].


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[17] S.K. Ashok, S. Murthy and J. Troost, *D-branes in non-critical superstrings and minimal super Yang-Mills in various dimensions*, *Nucl. Phys.* **B 749** (2006) 172 [hep-th/0504079].

[18] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici and A. Schwimmer, *Brane dynamics and $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 505** (1997) 202 [hep-th/9704104].

[19] N.J. Evans, C.V. Johnson and A.D. Shapere, *Orientifolds, branes and duality of 4D gauge theories*, *Nucl. Phys.* **B 505** (1997) 251 [hep-th/9703210].

[20] M. Bershadsky, C. Vafa and V. Sadov, *D-strings on d-manifolds*, *Nucl. Phys.* **B 463** (1996) 398 [hep-th/9510225].

[21] H. Ooguri and C. Vafa, *Geometry of $N = 1$ dualities in four dimensions*, *Nucl. Phys.* **B 500** (1997) 62 [hep-th/9702180].

[22] K. Hori, K. Hosomichi, D.C. Page, R. Rabadán and J. Walcher, *Non-perturbative orientifold transitions at the conifold*, *JHEP* **10** (2005) 026 [hep-th/0506234].

[23] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of $G_2$ holonomy*, *Adv. Theor. Math. Phys.* **6** (2003) 1 [hep-th/0107177].

[24] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B 463** (1996) 55 [hep-th/9511164];

[25] S. Mukhi and C. Vafa, *Two-dimensional black hole as a topological coset model of $c = 1$ string theory*, *Nucl. Phys.* **B 407** (1993) 667 [hep-th/9301083].

[26] D. Ghoshal and C. Vafa, *$c = 1$ string as the topological theory of the conifold*, *Nucl. Phys.* **B 453** (1995) 121 [hep-th/9506122].

[27] S.K. Ashok, S. Murthy and J. Troost, *Topological cigar and the $c = 1$ string: open and closed*, *JHEP* **02** (2006) 013 [hep-th/0511239].

[28] J. Teschner, *Remarks on Liouville theory with boundary*, hep-th/0009138.

[29] F. Cachazo, B. Fiol, K.A. Intriligator, S. Katz and C. Vafa, *A geometric unification of dualities*, *Nucl. Phys.* **B 628** (2002) 3 [hep-th/0110028].

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